Here we first restate Theorem 1 and 2 in a more formal form, and then give proofs to them.

THEOREM 1. *Given the set $\mathcal{T}$ of local taxonomies, let $\mathbf{O} = O_1O_2...O_n$ be any operation sequence producing a taxonomy graph $G_{\mathbf{O}}$ s.t. i) each $O_i$ is either a horizonal or vertical merge operation on elements in $\mathcal{T}$ (for $1 \leq i \leq n$) and ii) no further operations could be performed. If $\mathbf{O}^\alpha$ and $\mathbf{O}^\beta$ are two different such sequences, then $G_{\mathbf{O}^\alpha}$ is the same as $G_{\mathbf{O}^\beta}$.*

PROOF. Suppose that after $\mathbf{O}^\alpha$ and $\mathbf{O}^\beta$, the resulting set of local taxonomies is $\mathcal{T}^\alpha$ and $\mathcal{T}^\beta$, respectively.

(I) We first show that $\mathcal{T}^\alpha = \mathcal{T}^\beta$. Since the set of local taxonomies is only affected by horizontal merge operations, and each horizontal merge operation only affects local taxonomies with the same root label, it is sufficient to show that for any two local taxonomies $T_1$ and $T_2$ in the original $\mathcal{T}$ such that $T_1$ and $T_2$ are rooted at $x^i$ and $x^j$ respectively, $T_1$ and $T_2$ are merged after $\mathbf{O}^\alpha$ if and only if they are merged by $\mathbf{O}^\beta$.

If $T_1$ and $T_2$ are finally merged after $\mathbf{O}^\alpha$, then either 1) $Child(T_1)$ and $Child(T_2)$ are similar, or 2) after certain operation $O_i^\alpha$, there exist some $T_1'$ with root $x^i$ and $T_2'$ with root $x^j$ such that $Child(T_1) \subseteq Child(T_1')$, $Child(T_2) \subseteq Child(T_2')$, and $Child(T_1')$ and $Child(T_2')$ are similar.

In case 1), $T_1$ and $T_2$ will also be merged after $\mathbf{O}^\beta$. We can prove this by contradiction. Suppose that $T_1$ and $T_2$ are not merged after $\mathbf{O}^\beta$, then let $T_1^\beta$ and $T_2^\beta$ be the local taxonomies in $\mathcal{T}^\beta$ with root $x^i$ and $x^j$ such that $Child(T_1) \subseteq Child(T_1^\beta)$ and $Child(T_2) \subseteq Child(T_2^\beta)$. Since $Child(T_1)$ and $Child(T_2)$ are similar, then according to Property 4, we have $Child(T_1^\beta)$ and $Child(T_2^\beta)$ are similar. As a result, $T_1^\beta$ and $T_2^\beta$ could be merged by appending another horizontal merge operation after $\mathbf{O}^\beta$, which contradicts to the definition of $\mathbf{O}^\beta$.

In case 2), we can prove that $Child(T_1') \subseteq Child(T_1^\beta)$ and $Child(T_2') \subseteq Child(T_2^\beta)$ by contradiction. Consider the situation if $Child(T_1') \not\subseteq Child(T_1^\beta)$. Let the horizontal merge operations that lead $T_1$ to $T_1'$ be $O_{i_1}^\alpha$, ..., $O_{i_k}^\alpha$. Without loss of generality, we could suppose there is some $1 \leq j \leq k$ such that the local taxonomy $T_1^j$ after $O_{i_j}^\alpha$ is the first one such that $Child(T_1^j) \not\subseteq Child(T_1^\beta)$ (so $Child(T_1^{j-1}) \subseteq Child(T_1^\beta)$). Assume we merge $T_1^{j-1}$ (let $T_1^0 = T_1$) and another local taxonomy $T'$ in the operation $O_{i_j}^\alpha$, then $Child(T')$ and $Child(T_1^{j-1})$ are similar. Property 4 then implies that $Child(T')$ and $Child(T_1^\beta)$ are also similar, which means they could be merged by appending another horizontal merge operation after $\mathbf{O}^\beta$, a contradiction. Therefore we must have $Child(T_1^j) \subseteq Child(T_1^\beta)$, and by applying the same reasoning with induction on $j$, we could conclude that $Child(T_1^k) \subseteq Child(T_1^\beta)$, namely $Child(T_1') \subseteq Child(T_1^\beta)$. Similarly, we can prove that $Child(T_2') \subseteq Child(T_2^\beta)$. But since $Child(T_1')$ and $Child(T_2')$ are similar, again according to Property 4, we have $Child(T_1^\beta)$ and $Child(T_2^\beta)$ are similar, which implies that $T_1^\beta$ and $T_2^\beta$ could be further merged, a contradiction again.

Therefore, if $T_1$ and $T_2$ are merged after $\mathbf{O}^\alpha$, they should also be merged after $\mathbf{O}^\beta$. The other direction could be proved exactly the same, by interchanging the superscripts of $\alpha$ and $\beta$. Hence we could now conclude that $\mathcal{T}^\alpha = \mathcal{T}^\beta$.

(II) Our next goal is to show that each link in $G_{\mathbf{O}^\alpha}$ between roots of local taxonomies in $\mathcal{T}^\alpha$ should also appear in $G_{\mathbf{O}^\beta}$, between the same two elements in $\mathcal{T}^\beta$, and vice versa. Suppose $(x^i, y^m)$ is such a link in $G_{\mathbf{O}^\alpha}$, and let $T_3$ and $T_4$ be the corresponding local taxonomies in $\mathcal{T}^\beta$ with root $x^i$ and $y^m$, respectively, with $y \in Child(T_3)$. Since $T_3$ and $T_4$ are both merged from a set of original local taxonomies in $\mathcal{T}$, there is some vertical merge operation to link $T_3'$ and $T_4'$ with root $x^i$ and $y^m$, where $Child(T_3') \subseteq Child(T_3)$, $Child(T_4') \subseteq Child(T_4)$, and $Child(T_3')$ and $Child(T_4')$ are similar, with $y \in Child(T_3')$. But then, according to Property 4, $Child(T_3)$ and $Child(T_4)$ are similar. Since both $T_3$ and $T_4$ are within $\mathcal{T}^\beta$ as well, they must have been linked. Otherwise we can now append an additional vertical merge operation after $\mathbf{O}^\beta$ to link them, which is a contradiction. Again, the other direction could be proved exactly in the same way, by interchanging the superscripts of $\alpha$ and $\beta$.

Since the resulting graph $G_{\mathbf{O}^\alpha}$ and $G_{\mathbf{O}^\beta}$ are nothing more than a set of local taxonomies and the interlinks between their roots, we conclude that $G_{\mathbf{O}^\alpha}$ and $G_{\mathbf{O}^\beta}$ are the same by combining (I) and (II). $\square$

THEOREM 2. *Let the set of all possible operation sequences be $\mathcal{O}$, and let $M = \min\{|\mathbf{O}| : \mathbf{O} \in \mathcal{O}\}$. Suppose $\mathbf{O}^\sigma$ is the operation sequence by first performing all possible horizontal merges and then all possible vertical merges, then $|\mathbf{O}^\sigma| = M$.*

PROOF. We prove the theorem in two steps. In the following, given an operation sequence $\mathbf{O}$, we use $H_{\mathbf{O}}$ and $V_{\mathbf{O}}$ to denote the set of all the horizontal and vertical merge operations within $\mathbf{O}$, respectively.

(I) First, we will show that, for any two operation sequences $\mathbf{O}^\alpha$ and $\mathbf{O}^\beta$, $|H_{\mathbf{O}^\alpha}| = |H_{\mathbf{O}^\beta}|$. Note that, since each horizontal merge operation merges exactly two local taxonomies into one new larger local taxonomy, the number of local taxonies decreasese by exactly 1 after each horizontal merge operation. Since the original set $\mathcal{T}$ of local taxonomies before $\mathcal{O}^\alpha$ and $\mathcal{O}^\beta$ are the same, and according to Theorem 1, the resulting set $\mathcal{T}'$ of local taxonomies after $\mathcal{O}^\alpha$ and $\mathcal{O}^\beta$ are also the same, we hence have $|H_{\mathbf{O}^\alpha}| = |H_{\mathbf{O}^\beta}| = |\mathcal{T}'| - |\mathcal{T}|$.

(II) Next, we claim that for any vertical operation in $V_{\mathbf{O}^\sigma}$ which links two local taxonomies $T_1^\sigma$ and $T_2^\sigma$ in $\mathcal{T}^\sigma$, with roots $x^i$ and $y^m$ respectively, there will be at least one vertical merge operation within any operation sequence $\mathbf{O}$ that merges two local taxonomies $T_1$ and $T_2$, with roots $x^i$ and $y^m$, such that $Child(T_1) \subseteq Child(T_1^\sigma)$ and $Child(T_2) \subseteq Child(T_2^\sigma)$. This is also a natural result from Theorem 1. Since $T_1^\sigma$ and $T_2^\sigma$ will also occurs in the resulting set of local taxonomies after $\mathbf{O}$, and since $T_1^\sigma$ and $T_2^\sigma$ are linked in $G_{\mathbf{O}^\sigma}$, they should also be linked in $G_{\mathbf{O}}$. Therefore, there must be some vertical merge operations in $V_{\mathbf{O}}$ that does this merge. What's, each different link such link in $G_{\mathbf{O}^\sigma}$ will have a different counterpart in $G_{\mathbf{O}}$. Thus we can conclude that $|V_{\mathbf{O}^\sigma}| \leq |V_{\mathbf{O}}|$. Also note that we could not always have $|V_{\mathbf{O}^\sigma}| = |V_{\mathbf{O}}|$.

Based on (I) and (II), we have $|\mathbf{O}^\sigma| = |H_{\mathbf{O}^\sigma}| + |V_{\mathbf{O}^\sigma}| \leq |H_{\mathbf{O}}| + |V_{\mathbf{O}}| = |\mathbf{O}|$, for any operation sequence $\mathbf{O}$, which completes the proof of the theorem. $\square$